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The Bailey–Orowan equation

S. O. OJEDIRAN, O. AJAJA*
Department of Physics, University of Ife, Ile-Ife, Nigeria

A simple derivation of the Bailey–Orowan equation, $\dot{\varepsilon} = R/H$, which is based on the spurt-like glide of dislocations during recovery-creep, is presented. It is demonstrated that this equation is valid for steady state but not for transient creep. A dislocation network model is employed to show that the values of R and H which are measured by stress-change techniques do not represent the true values of the recovery and work-hardening rates. However, the ratio of the measured values is always equal to the strain rate during transient or steady state creep.

1. Introduction

In a recent paper [1] an issue was raised over the validity of the methods of derivation and experimental verification of the Bailey–Orowan equation

$$\dot{\varepsilon} = R/H \quad (1)$$

where $\dot{\varepsilon}$ is the strain rate during creep while R and H are the recovery and the work-hardening rates, respectively. The application of this equation to both transient and steady-state creep presupposes the existence of an equation of state involving the applied stress, σ, the creep strain, ε, and time, t. Such an equation of state, it was demonstrated, does not exist [1]. A method of derivation of the Bailey–Orowan equation is presented in this paper. The significance of the parameters R and H, and the experimental methods of their determination are closely examined in the light of this new derivation.

2. Derivation of the Bailey–Orowan equation

During recovery-creep, dislocations are often believed to glide in a "jerky" fashion, i.e. a dislocation spends a relatively long time waiting at an obstacle and, upon overcoming the obstacle glides rapidly to the next obstacle. For this spurt-like dislocation motion the strain $\Delta\varepsilon$ generated in time Δt when density $\Delta\varrho_m$ of dislocations are mobilized can be written as [2, 3]

$$\Delta\varepsilon = \alpha_1 \Delta\varrho_m \boldsymbol{b} L \quad (2)$$

where $\boldsymbol{b}$ is the Burgers vector of the dislocations, α_1 is a constant and L is closely related to the obstacle spacing.

Assuming that dislocation generation occurs each time a released dislocation link glides and increases its length in the process, the density of dislocations generated ($\Delta\varrho_g$) can be taken to be directly proportional to the density mobilized, i.e.

$$\Delta\varrho_g = \beta\Delta\varrho_m \quad (3)$$

where β is a proportionality constant. The strain rate is given by Equation 2, which, combined with Equation 3 yields, in the limit $\Delta t \to 0$

$$\dot{\varepsilon} = \alpha_2 \dot{\varrho}_g \boldsymbol{b} L \quad (4)$$

where $\alpha_2 = \alpha_1/\beta$. At steady-state the rates of dislocation generation ($\dot{\varrho}_g$) and annihilation ($\dot{\varrho}_a$) must be equal, whereupon Equation 4 becomes

$$\dot{\varepsilon}_s = \alpha_2 \dot{\varrho}_a \boldsymbol{b} L \quad (5)$$

where the subscript s represents steady-state.

During classical work-hardening in the absence of concomitant recovery, the flow stress, σ_f, is usually related to the dislocation density, ϱ, according to the equation

$$\sigma_f = \alpha_0 G \boldsymbol{b} \varrho^{1/2} \quad (6)$$

where G is the shear modulus of rigidity of the material and α_0 is a constant. The recovery and work-hardening rates are defined as

$$R = -(\partial\sigma_f/\partial t)_\varepsilon \quad (7)$$

and

$$H = (\delta\sigma_f/\delta\varepsilon)_t \quad (8)$$

respectively. If a previously deformed sample is subjected to static recovery at high temperature for time δt such that the dislocation density changes by $\delta\varrho|_\varepsilon$, the change in flow stress upon re-testing at low temperature is, from Equations 6 and 7,

$$\delta\sigma_f = -R\delta t = \frac{\alpha_0 G \boldsymbol{b} \delta\varrho|_\varepsilon}{2\varrho^{1/2}}$$

which gives, as $\delta t \to 0$,

$$R = \alpha_0 G \boldsymbol{b} \dot{\varrho}_a/(2\varrho^{1/2}) \quad (9)$$

where $\dot{\varrho}_a = -(\delta\varrho/\delta t)_\varepsilon$. Similarly, the change in flow stress when a previously deformed sample is further deformed by $\delta\varepsilon$ in the absence of recovery such that density $\delta\varrho_g$ of dislocations is generated is (Equations 6 and 8):

$$\delta\sigma_f = H\delta\varepsilon = \alpha_0 G \boldsymbol{b} \delta\varrho_g/(2\varrho^{1/2})$$

or, as $\delta\varepsilon \to 0$,

$$H = \frac{\alpha_0 G \boldsymbol{b}}{2\varrho^{1/2}} \frac{d\varrho_g}{d\varepsilon}. \quad (10)$$

*Author to whom all correspondence should be addressed.

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Noting that $d\varrho_g/d\varepsilon = 1/(\alpha_2 \boldsymbol{b} L)$ from Equation 4, Equation 10 becomes

$$H = \alpha_0 G/(2\alpha_2 \varrho^{1/2} L) \qquad (11)$$

Combining Equation 11 with Equation 9 yields

$$\frac{R}{H} = \alpha_2 \dot{\varrho}_a \boldsymbol{b} L \qquad (12)$$

which, when compared with Equation 5 gives, at steady-state,

$$\dot{\varepsilon}_s = R/H \qquad (13)$$

It becomes immediately clear that even though Equation 12 is always true, Equation 13, or the Bailey–Orowan equation, is valid only at steady-state. Equation 13 is made possible by the equality between the dislocation generation and annihilation rates at steady-state. This, in essence, tallies with Bailey and Orowan's original conception of steady-state reflecting a balance between the two competing processes of work-hardening and recovery [4, 5].

It should be noted that Equations 9 and 11 give the true values of the recovery and work-hardening rates in the sense that their derivation does not depend on any particular deformation model. The use of Equation 6 is quite legitimate because this equation is expected to hold valid for each individual step of recovery and glide provided, of course, that flow stress changes during the low-temperature deformation are ascribable only to changes in dislocation density. Equation 4, the only other relationship used in the derivation, is applicable whenever dislocation glide is spurt-like, as is believed to be the case for pure metals and some solid solution alloys.

3. The strain rate equation for recovery creep

A recently proposed dislocation network model [3] will be used to examine the significance of the experimentally determined parameters which are often regarded as the work hardening and recovery rates. The model considers a distribution function $\phi(\lambda, t)$ such that $\phi(\lambda, t)d\lambda$ represents the number of dislocation links per unit volume having lengths between λ and $\lambda + d\lambda$ (Fig. 1). The average link size is $\langle\lambda\rangle$ while λ_a is the threshold link size which favourably oriented links have to attain before they can glide. These can be expressed as

$$\langle\lambda\rangle = \alpha_3 \varrho^{-1/2} \qquad (14)$$

and

$$\lambda_a = \alpha_4 G\boldsymbol{b}/\sigma \qquad (15)$$

where α_3 and α_4 are constants. For recovery-creep the density of dislocations which are usually gliding at any instant is very small, which means that λ_a must lie close to the right tail end of the distribution [6].

The strain rate derived from this model is [3]

$$\dot{\varepsilon} = \alpha_1 \psi(t) \dot{\varrho}_a \boldsymbol{b} L \qquad (16)$$

where

$$\psi(t) = \tfrac{1}{2} f_p \alpha_3 \alpha_5 \phi_a \lambda_a \varrho^{-3/2} \qquad (17)$$

In Equation 17, f_p is the fraction of links which are favourably oriented for glide, $\phi_a = \phi(\lambda_a, t)$ and α_5 is the ratio of the growth rate (during recovery) of links of length λ_a to the growth rate of the average link size, $\langle\lambda\rangle$. Thus

$$\frac{d\lambda_{a,r}}{dt} = \alpha_5 \frac{d\langle\lambda\rangle}{dt} \qquad (18)$$

where the subscript r has been included to indicate that the change in λ_a is due to recovery.

4. Evaluation of stress change tests

Stress change tests are frequently used for the determination of R and H, as well as the verification of the Bailey–Orowan equation. In the stress reduction method of determining R, the stress on a creeping sample is reduced by a small amount ($\Delta\sigma$). If creep recommences after time Δt_r, R is taken as $\Delta\sigma/\Delta t_r$ as $\Delta\sigma \to 0$ [7, 8].

If the stress, σ, on a sample creeping at steady-state is reduced by $\Delta\sigma$, all links which happen to be gliding at the moment of stress interruption will almost instantly become arrested at the nearest obstacles. Thus no plastic strain should be observed until links of size λ_a would have grown by recovery to the new threshold size $\lambda_a + \Delta\lambda_{a,\sigma}$ corresponding to the new stress $\sigma - \Delta\sigma$ (Fig. 2). Within this (incubation) time, Δt_r, required for creep to recommence, the whole network would have coarsened such that the average link size increases to $\langle\lambda\rangle + \Delta\langle\lambda\rangle$, corresponding to the frequency function $\phi_{\sigma-\Delta\sigma}$. From Equation 18, the increase in λ_a due to recovery is

$$\Delta\lambda_{a,r} = \alpha_5 \Delta\langle\lambda\rangle \qquad (18a)$$

where α_5 will be expected to depend on the details of the network geometry such as the critical link size required for the growth of individual links as well as the values of $\langle\lambda\rangle$ and λ_a.

From Equation 14, $\Delta\langle\lambda\rangle = -\alpha_3\varrho^{-3/2}\Delta\varrho/2 = \alpha_3\varrho^{-3/2}\dot{\varrho}_a\Delta t_r/2$ in the limit $\Delta t_r \to 0$, where $\Delta\varrho = -\dot{\varrho}_a\Delta t_r$. Combining this with Equation 18a gives

$$\Delta\lambda_{a,r} = \alpha_3\alpha_5\varrho^{-3/2}\dot{\varrho}_a\Delta t_r/2 \qquad (19)$$

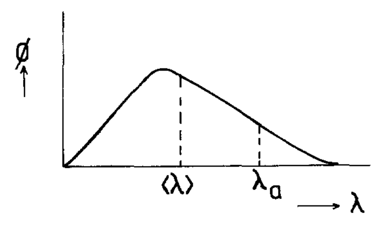

Figure 1 Schematic illustration of the distribution function $\phi(\lambda, t)$.

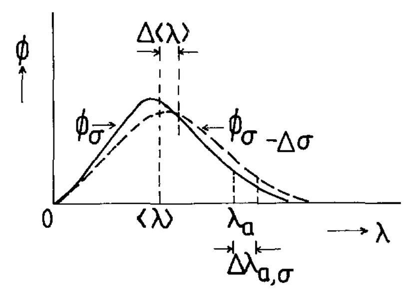

Figure 2 Schematic illustration of the distribution functions before and after a stress reduction.

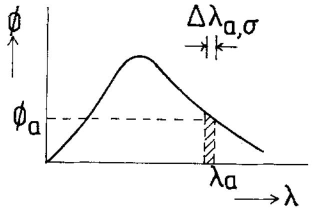

Figure 3 Illustration of the number of links mobilized upon increasing the stress by $\Delta\sigma$ (shaded area).

In the limit $\Delta\sigma \to 0$, the increase in threshold size corresponding to a stress reduction, $\Delta\sigma$, is found from Equation 15

$$\Delta\lambda_{a,\sigma} = -\alpha_4 G\boldsymbol{b}\Delta\sigma/\sigma^2 \tag{20}$$

For creep to recommence after time Δt_r, $\Delta\lambda_{a,r}$ must be equal to $\Delta\lambda_{a,\sigma}$. Using the expression for R in Equation 9, we can combine Equations 19 and 20 to give

$$R_m = -\frac{\Delta\sigma}{\Delta t_r} = \left[\frac{\alpha_3\alpha_5\sigma^2}{\alpha_0\alpha_4 G^2\boldsymbol{b}^2\varrho}\right] R \tag{21}$$

where R_m is the measured parameter which is often taken to be the recovery rate. Noting that $\sigma = \alpha_0 G\boldsymbol{b}\varrho_s^{1/2}$ at steady-state, Equation 21 can be written as

$$R_m = \left[\left(\frac{\alpha_0\alpha_3\alpha_5}{\alpha_4}\right)\left(\frac{\varrho_s}{\varrho}\right)\right] R \tag{22}$$

which is valid during transient or steady-state creep. At steady-state $\varrho = \varrho_s$ and R_m then becomes

$$R_{m,s} = \frac{\alpha_0\alpha_3\alpha_5}{\alpha_4} R \tag{23}$$

As will be discussed later, Equations 22 and 23 show that the measured "recovery rate" is clearly different from, albeit closely related to, the true recovery rate.

To determine H, the stress on a creeping sample is increased by $\Delta\sigma$, whereupon an instantaneous plastic strain $\Delta\varepsilon$ results. H is then taken as $\Delta\sigma/\Delta\varepsilon$ in the limit $\Delta\sigma \to 0$ [8–10]. Considering the distribution function in Fig. 3, the threshold size λ_a decreases by $\Delta\lambda_{a,\sigma}$ upon stress increase, in accordance with Equation 20. The density of dislocations which are mobilized instantaneously is equal to (shaded area in Fig. 3) $\phi_a\lambda_a\Delta\lambda_{a,\sigma}$, so that the strain ($\Delta\varepsilon$) is (Equation 2)

$$\Delta\varepsilon = -\alpha_1\boldsymbol{b}Lf_p\phi_a\lambda_a\Delta\lambda_{a,\sigma} \tag{24}$$

where f_p and λ_a are as defined earlier. Combining Equations 20 and 24 gives

$$H_m = \frac{\Delta\sigma}{\Delta\varepsilon} = \frac{\sigma^2}{\alpha_1\alpha_4 f_p\phi_a\lambda_a G\boldsymbol{b}^2 L} \tag{25}$$

where H_m is the measured "work-hardening rate". Noting again that $\sigma = \alpha_0 G\boldsymbol{b}\varrho_s^{1/2}$ and using the expression for H in Equation 11, Equation 25 becomes

$$H_m = \left[\frac{\alpha_0\alpha_3\alpha_5}{\alpha_4}\frac{\varrho_s}{\varrho}\right]\left(\frac{H}{\beta\psi(t)}\right) \tag{26}$$

where $\psi(t)$ is as defined in Equation 17, $L \cong \varrho^{-1/2}$ and $\alpha_2 = \alpha_1/\beta$. By comparing Equations 5 and 16 it is clear that at steady-state, $\beta\psi(t) = 1$, whereupon Equation 26 gives

$$H_{m,s} = \frac{\alpha_0\alpha_3\alpha_5}{\alpha_4} H \tag{27}$$

Equations 26 and 27 show that H_m is not identical to H at any stage of creep.

By virtue of Equation 13, we can combine Equations 23 and 27 to yield, at steady-state

$$\frac{R_{m,s}}{H_{m,s}} = \frac{R}{H} = \dot{\varepsilon}_s \tag{28}$$

During transient creep, Equations 22 and 26 give

$$\frac{R_m}{H_m} = \beta\psi(t)\frac{R}{H} \tag{29}$$

which, combined with Equation 12 gives

$$\frac{R_m}{H_m} = \alpha_1\psi(t)\dot{\varrho}_a\boldsymbol{b}L \tag{30}$$

The right-hand side is again equal to the strain rate (Equation 16). It thus becomes quite clear that even though the measured values of R and H differ from the true values, their ratio always gives the correct value of the creep rate during transient or steady-state creep, i.e.

$$\dot{\varepsilon} = \frac{R_m}{H_m} \tag{31}$$

In terms of the true values of R and H, the creep rate during transient creep is (Equations 29 and 31):

$$\dot{\varepsilon} = \beta\psi(t)\frac{R}{H} \tag{32}$$

Equation 32 clearly shows that the Bailey–Orowan equation is not applicable to transient creep.

5. Conclusions

The observed agreement between R_m/H_m and $\dot{\varepsilon}$ is really not surprising. What the stress change tests do is simply break the creep process into the two constituent steps of glide and recovery. Stress reduction ($-\Delta\sigma$) during creep essentially suppresses strain generation for a time (Δt_r) during which some links can grow past the threshold size characteristic of the (original) creep stress. Thereafter, an increase in stress ($+\Delta\sigma$) results in an instantaneous plastic strain ($\Delta\varepsilon$) due to the glide of these newly generated "mobile" links. In the limit $\Delta\sigma \to 0$ a cycle of alternate stress increase and reduction becomes identical to a (constant stress) creep test, and the strain rate is equal to the strain generated during the cycle ($\Delta\varepsilon$) divided by the elapsed time (Δt_r), i.e.

$$\dot{\varepsilon} = \frac{\Delta\varepsilon}{\Delta t_r} = \frac{\Delta\sigma/\Delta t_r}{\Delta\sigma/\Delta\varepsilon} = \frac{R_m}{H_m}$$

In concluding, it is emphasized that the quantities R_m and H_m as determined from stress-change tests do not represent the true values of the recovery and work-hardening rates. At steady-state both R_m and H_m differ from R and H by the factor $\alpha_0\alpha_3\alpha_5/\alpha_4$ (Equations 23 and 27). Although each of the parameters α_0, α_3 and α_4 is expected to be of the order of unity, the value of α_5 could be much larger than one, depending on the kinetics of network coarsening. In such a case appreciably overestimated values of R and H will result. The fact that the (measured) values of R_m have been close to the relaxation rate [11, 12] while those of H_m have been of the order of the elastic modulus [10, 13] is evidence of their gross over-estimation.

From Equation 11, $H = (\alpha_0/2\alpha_2)G = (\alpha_0\beta/2\alpha_1)G$ if we take $L \simeq \varrho^{-1/2}$. As α_0 and α_1 are close to unity, the usual observation that H is orders of magnitude lower than G during low-temperature deformation implies that $\beta \ll 1$. Furthermore, H should be independent of ϱ, as is actually the case during the stage II deformation of single crystals. The observation that H_m increases during transient creep [8, 9] could be ascribed to the time dependences of ϱ and $\psi(t)$ in Equation 26.

References

1. O. AJAJA, *Res Mechanica* **10** (1984) 79.
2. H. MECKING and K. LUCKE, *Scripta Metall.* **4** (1970) 427.
3. O. AJAJA, *J. Mater. Sci.* **21** (1986) 3351.
4. R. W. BAILEY, *J. Inst. Metals* **35** (1926) 27.
5. E. OROWAN, *J. West Scotland Iron Steel Inst.* **54** (1946–47) 45.
6. J. B. FAGBULU and O. AJAJA, *J. Mater. Sci. Lett.* **6** (1987) 894.
7. S. K. MITRA and D. McLEAN, *Proc. Roy. Soc. A* **295** (1966) 288.
8. W. J. EVANS and B. WILSHIRE, *Trans. Met. Soc. AIME* **242** (1968) 2514.
9. D. SIDEY and B. WILSHIRE, *Metal Sci. J.* **3** (1969) 56.
10. H. OIKAWA and M. OHNUMA, *Met. Trans.* **12A** (1981) 1699.
11. T. WATANABE and S. KARASHIMA, *Trans. Jpn Inst. Metals* **9** (suppl.) (1968) 242.
12. *Idem, Metal Sci. J.* **4** (1970) 52.
13. H. OIKAWA and K. SUGAWARA, *Scripta Metall.* **12** (1978) 85.

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